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AN INCREMENTAL FORM OF THE SINGLE-INTEGRAL NONLINEAR VISCOELAST--ETC(U)  
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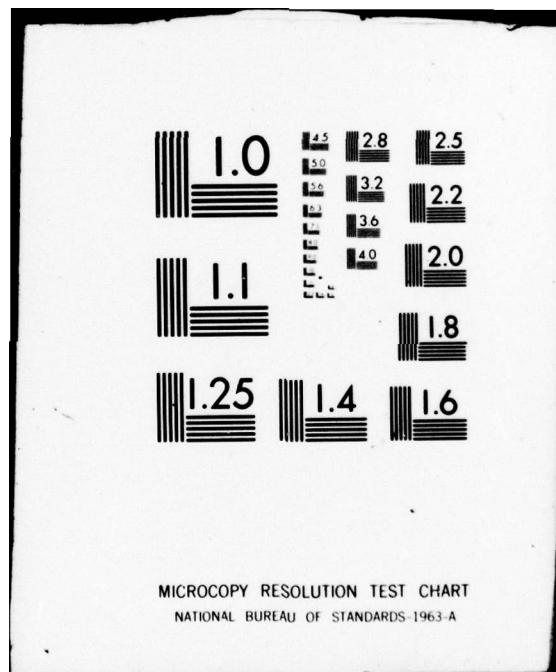
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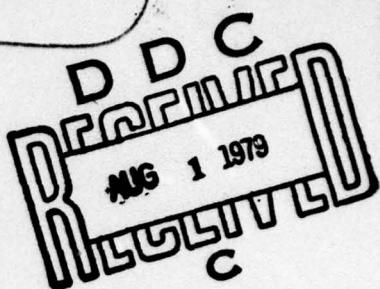
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AN INCREMENTAL FORM OF THE SINGLE-INTEGRAL  
NONLINEAR VISCOELASTIC THEORY FOR ELASTIC-PLASTIC-CREEP  
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# An Incremental Form of the Single-Integral Nonlinear Viscoelastic Theory for Elastic-Plastic-Creep Finite Element Analysis

D. R. SANDERS

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## ABSTRACT

→ A single-integral nonlinear viscoelastic theory is cast into a generalized incremental form for use in predicting creep response of materials. The incremental creep constitutive model is tailored for use in an incremental finite element program wherein the equations of motion are integrated numerically step-by-step. Recursive relations for material memory parameters are developed which allow the retention of all past history prior to the current time step and require integration over the current time step only. The creep model parameters required in the theory are obtainable from a standard creep and recovery test. The incorporation of the model within an incremental elastic-plastic finite element formulation is outlined.

## INTRODUCTION

In a majority of the finite element programs in use today for analyzing metal components, the traditional approach of separating the total strain into elastic, plastic, and creep components is used (1,2,3). The most widely accepted means of predicting the creep strain in this context is the phenomenological creep theory (4,5). As an alternative to this approach, a single integral nonlinear viscoelastic theory has been obtained by Schapery (6) from thermodynamic considerations. It has a form similar to the Boltzman integral used in linear viscoelasticity, but the physical time is replaced by a so called reduced time; and several other material functions enter into the theory. In theory, this representation of the inelastic strain components should provide, within the same constitutive theory, the capability of analyzing both polymeric and metallic materials.

Two constitutive theories similar in appearance to the single integral theory of Schapery's are given

by Rashid (7) and Valanis (8). Rashid uses a modified superposition principle with the compliance, which is a function of stress and time, transformed such that it is a function of a single parameter called reduced time. Valanis's theory (8) is derived from thermodynamic considerations with the resulting single integral equation appearing in terms of the strain history instead of the stress history as in reference (6). This theory is very similar to Schapery's theory (9) wherein strain is employed as the independent variable instead of stress. The functional difference in references (8) and (9) is due to their respective definitions of reduced time. The other material functions entering into references (6,9) are not present in Rashid's or Valanis's theory.

Application of the single integral theory has been made by Valanis (10,11) for the viscoplastic response of several metals. It has also been shown to give accurate results for several polymeric composites in creep, creep-recovery and multistep input tests on uniaxial specimens (12,13).

A potential advantage of the single integral approach is that it easily handles multistep input histories including stress reversal and creep-recovery. This is not true of the phenomenological theory of creep for metals based on strain hardening which must resort to special procedures to handle the stress reversal situation as outlined in (14). These procedures have in some cases resulted in rather poor results as indicated in reference (14). Further, the phenomenological theory has no predictive capability for the recovery response to stress removal. Thus, the single integral constitutive theory may offer additional predictive capabilities over the phenomenological theory of creep.

This paper casts the single integral theory into a form suitable for use within an incremental finite element program. Since the equations of motion are

integrated numerically step-by-step, the time derivative of the single integral equation must be obtained. In this form, the creep strain increment can be determined for the current time interval based on the creep strain rate at the beginning of the step. Computationally, the relationships derived for the rate form of the single integral equation are particularly appealing since recursive relationships are developed which allow for the retention of all past history prior to the current time step and require integration over the current time step only.

The incorporation of the model within an incremental elastic-plastic finite element formulation is outlined. The time-independent incremental plasticity relations with a von Mises yield criterion and a kinematic hardening rule are employed to describe the plastic component of strain.

A graphical procedure for determining the material functions from either creep data or creep equations is presented. It is shown that this procedure is only applicable to materials whose elastic strain change on unloading is equal to that on loading.

Numerical comparison of the single integral theory to that of phenomenological theory of creep are given for creep, creep and recovery, multistep and reverse loading histories for a type 304 stainless steel. The numerical results were obtained using a computer program MONVIS developed by the authors which implements the integration procedure developed below for the single integral creep equation.

It is suggested that the rate form of the single-integral equations appearing in (7,8,9) could be integrated by employing a procedure similar to that given in this paper for the single integral constitutive theory of reference (6), provided the kernel function is a combination of constants, linear, and exponential terms.

#### RATE FORM OF THE SINGLE INTEGRAL EQUATION

The single integral nonlinear viscoelastic constitutive equation derived in reference (6) for an initially isotropic material is written for isothermal conditions, incompressible creep strains, and infinitesimal strains as,

$$\dot{\epsilon}_{ab} = J(0)\dot{\sigma}_{ab} + \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} \int_{-\infty}^{\psi} \Delta J(\psi-\psi') \frac{dG_{cd}}{d\psi'} d\psi' \quad (1)$$

$$\psi = \int_0^t \frac{a_G}{a_D} dt, \quad (2a)$$

$$G_{cd} = \frac{\hat{\sigma}_{cd}}{a_G}, \quad (2b)$$

$$\Delta J(\psi) = \sum_r A(r) (1 - e^{-\psi/\tau_r}) + \sum_{r'} B(r') \psi, \quad (2c)$$

and  $A(r)$ ,  $B(r')$  and  $\tau_r$  are constants. The functions  $a_G$ ,  $a_D$  and  $\hat{\sigma}_{cd}$  are material properties and are in general functions of the three stress invariants. The quantity  $\psi$  is a reduced time variable while  $\dot{\epsilon}_{ab}$  are engineering strain components and  $\sigma_{cd}$  are components of the Cauchy stress tensor. The function  $\Delta J(\psi)$  is the transient component of the so called linear viscoelastic shear creep compliance. The first term in equation (1) represents the instantaneous elastic response of the material.

To cast equation (1) into a form suitable for use in an incremental finite element analysis, its time derivative is taken, and equation (2a) is used to obtain,

$$\dot{\epsilon}_{ab} = J(0)\dot{\sigma}_{ab} + \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} \int_{-0}^t \Delta J(\psi-\psi') \frac{dG_{cd}}{d\psi'} dt' + \frac{a_D}{a_G} \left( \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} \right) \int_{-0}^t \frac{d(\Delta J(\psi-\psi'))}{d\psi} \frac{dG_{cd}}{d\psi'} dt'. \quad (3)$$

Equation (3) is rewritten as,

$$\dot{\epsilon}_{ab} = J(0)\dot{\sigma}_{ab} + \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} X_{cd} + \frac{a_D}{a_G} \left( \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} \right) Y_{cd}, \quad (4)$$

where

$$X_{cd} = \int_{-0}^t \Delta J(\psi-\psi') \frac{dG_{cd}}{d\psi'} dt', \quad (5)$$

$$Y_{cd} = \int_{-0}^t \Delta J'(\psi-\psi') \frac{dG_{cd}}{d\psi'} dt', \quad (6)$$

and

$$\Delta J'(\psi-\psi') = \frac{d \Delta J(\psi-\psi')}{d\psi}. \quad (7)$$

A time marching procedure for evaluating  $X_{cd}$  and  $Y_{cd}$  is formulated below.

#### Evaluation of $X_{cd}$ and $Y_{cd}$

The evaluation of the two integrals  $X_{cd}$  and  $Y_{cd}$  given by equations (5) and (6) represents a key element in evaluating the creep strain rate. Once the value of these integrals are known, the creep strain rate is evaluated from the sum of the second and third terms of equation (4) as:

$$\dot{\epsilon}_{ab}^c = \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} X_{cd} + \frac{a_D}{a_G} \left( \frac{\partial \hat{\sigma}_{cd}}{\partial \sigma_{ab}} \right) Y_{cd}. \quad (8)$$

Two important features in the evaluation of  $X_{cd}$  and  $Y_{cd}$  are the determination of recursive formulas which make use of values from the previous time step and require the evaluation of the two integrals over the current time step only. The formulation presented here is similar to the integration technique employed by Zak (15) and Bills, et al. (16) for a thermorheological simple material due to a variable temperature field.

Assume the time from 0 to  $t$  is subdivided into  $N$  subintervals such that  $t_0=0, t_1=t_0+\Delta t, \dots, t_N=t_{N-1}+\Delta t_N=t$ . For  $\sigma_{cd}(x, t_N)$  and  $\epsilon_{cd}(x, t_N)$ , write  $\sigma_{cdN}$  and  $\epsilon_{cdN}$  and assume that  $\sigma_{cdN}(0, t_N) = 0$  and  $\epsilon_{cdN}(0, t_N) = 0$ . Then  $X_{cd}$  and  $Y_{cd}$  can be written as

$$X_{cdN} = \int_0^{t_N} \Delta J(\psi_N - \psi) \frac{dG_{cd}}{d\psi} dt' \quad (9)$$

and

$$Y_{cd_N} = \int_0^{t_N} \Delta J'(\psi - \psi') \frac{dG_{cd}}{dt'} dt' . \quad (10)$$

Evaluation of  $X_{cd_N}$ . Using equation (2c) in equation (9), one obtains

$$X_{cd_N} = \int_0^{t_N} \sum_r A(r) \frac{dG_{cd}}{dt'} dt' \quad (11)$$

$$- \int_0^{t_N} \sum_r A(r) e^{-(\psi_N - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt'$$

$$+ \int_0^{t_N} \sum_{r'} B(r') (\psi_N - \psi') \frac{dG_{cd}}{dt'} dt' .$$

By evaluating the first integral in equation (11), expanding the second and third as the sum of two integrals and approximating the time rate of change of  $G_{cd}$  within the time interval  $t_{N-1} < t < t_N$  as

$$\frac{dG_{cd}}{dt} = \frac{G_{cd_N} - G_{cd_{N-1}}}{\Delta t_N} , \quad (12)$$

then equation (11) can be written as

$$X_{cd_N} = \sum_r A(r) G_{cd_N}$$

$$- \left[ \int_0^{t_{N-1}} \sum_r A(r) e^{-(\psi_N - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' \right. \quad (13)$$

$$- \left( \frac{G_{cd_N} - G_{cd_{N-1}}}{\Delta t_N} \right) \times$$

$$\left. \int_{t_{N-1}}^{t_N} \sum_r A(r) e^{-(\psi_N - \psi')/\tau_r} dt' \right]$$

$$+ \left[ \int_0^{t_{N-1}} \sum_{r'} B(r') (\psi_N - \psi') \frac{dG_{cd}}{dt'} dt' \right. \quad (13)$$

$$+ \left. \frac{G_{cd_N} - G_{cd_{N-1}}}{\Delta t_N} \int_{t_{N-1}}^{t_N} \sum_{r'} B(r') (\psi_N - \psi') dt' \right] .$$

Define the following quantities,

$$u1_N = \sum_r A(r) J1_N(r) , \quad (14)$$

$$u2_N = \sum_{r'} B(r') J2_N(r') , \quad (15)$$

$$u_N = u1_N - u2_N . \quad (16)$$

where,

$$J1_N(r) = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} e^{-(\psi_N - \psi')/\tau_r} dt' \quad (17)$$

and

$$J2_N(r') = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} (\psi_N - \psi') dt' . \quad (18)$$

With the use of equation (16), equation (13) is rewritten as

$$X_{cd_N} = \sum_r A(r) G_{cd_N} - G_{cd_N}(u_N) + L_{cd_N} \quad (19)$$

where,

$$L_{cd_N} = G_{cd_{N-1}}(u_N) - u1_{cd_N} + u2_N . \quad (20)$$

and

$$u1_{cd_N} = \sum_r A(r) C1_{cd_N}(r) , \quad (21)$$

$$u2_{cd_N} = \sum_{r'} B(r') C2_{cd_N}(r') , \quad (22)$$

$$C1_{cd_N}(r) = \int_0^{t_{N-1}} e^{-(\psi_N - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' , \text{ and} \quad (23)$$

$$C2_{cd_N}(r') = \int_0^{t_{N-1}} (\psi_N - \psi') \frac{dG_{cd}}{dt'} dt' . \quad (24)$$

Recursive relationships for equations (23) and (24) are developed in the Appendix and are given below as:

$$C1_{cd_N}(r) = e^{-\tau_r \Delta \psi_N} \left\{ C1_{cd_{N-1}}(r) + (G_{cd_{N-1}} - G_{cd_{N-2}}) J1_{N-1}(r) \right\} \quad (25)$$

and

$$C2_{cd_N}(r') = \Delta \psi_N G_{cd_{N-1}} + C2_{cd_{N-1}}(r') + (G_{cd_{N-1}} - G_{cd_{N-2}}) \times$$

$$J2_{N-1}(r') \quad (26)$$

where  $C1_{cd_{N-1}}(r)$ ,  $C2_{cd_{N-1}}(r)$ ,  $J1_{N-1}(r)$  and  $J2_{N-1}(r')$  are known from the previous time step. Thus, equation (19) can be evaluated by integrating  $J1_{cd_N}(r)$  and  $J2_{cd_N}(r')$  defined by equations (17) and (18), respectively, over the current time step only. All history effects are contained in  $L_{cd_N}$  defined by equation (20). In the Appendix, equations (17) and (18) are numerically integrated using Gaussian quadrature. Equation (17) becomes:

$$J1_N(r) = \frac{\tau_r}{\Delta t_N} \left( 1 - e^{-\Delta \psi_N / \tau_r} \right) + J1_N^{*}(r) \quad (27)$$

where,

$$J1^*(r) = \frac{1}{2} \int_{-1}^1 [e^{-(\psi_N - \psi')/\tau_r} - e^{-(1-\tau)\Delta\psi_N/\tau_r}] d\tau \quad (28)$$

and

$$\Delta\psi_N = \psi_N - \psi_{N-1}.$$

Equation (18) becomes:

$$J2_N^*(r) = \psi_N - J1_N^*(r) \quad (29)$$

where,

$$J2_N^*(r) = \frac{1}{2} \int_{-1}^1 \psi' d\tau. \quad (30)$$

Evaluation of the reduced time integral given in equation (2a) is also presented in the Appendix.

Evaluation of  $\gamma_{cdN}$ . The integration of  $\gamma_{cdN}$  is carried out in a similar manner to that used for  $x_{cdN}$  and is outlined briefly here.  $\gamma_{cdN}$  was given by equation (10) as,

$$\gamma_{cdN} = \int_0^{t_N} \Delta\psi'(\psi_N - \psi') \frac{dG_{cd}}{dt} dt'$$

where,

$$\Delta\psi'(\psi_N - \psi') = \sum_r \frac{A(r)}{\tau_r} e^{-\psi/\tau_r} + \sum_r B(r') \quad (31)$$

Expanding equation (10) as the sum of two integrals, using equation (31) and approximating the time rate of change of  $G_{cd}$  given by equation (12); equation (10) is rewritten as,

$$\begin{aligned} \gamma_{cdN} &= \int_0^{t_{N-1}} \left[ \sum_r \frac{A(r)}{\tau_r} e^{-(\psi_N - \psi')/\tau_r} + \sum_r B(r') \right] \times \\ &\quad \frac{dG_{cd}}{dt'} + \left[ \frac{dG_{cdN} - dG_{cdN-1}}{\Delta t_N} \right] \times \\ &\quad \int_{t_{N-1}}^{t_N} \left[ \sum_r \frac{A(r)}{\tau_r} e^{-(\psi_N - \psi')/\tau_r} + \sum_r B(r') \right] dt' \end{aligned} \quad (32)$$

Define the following quantities,

$$n_N = \sum_r B(r') + \sum_r \frac{A(r)}{\tau_r} J1_N^*(r). \quad (33)$$

$$K_{cdN} = -G_{cdN-1} n_N + \gamma_{cdN} \quad (34)$$

and

$$\gamma_{cdN} = \sum_r \frac{A(r)}{\tau_r} F_{cdN}^*(r) + \sum_r B(r') G_{cdN-1}. \quad (35)$$

where

$$J1_N^*(r) = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} e^{-(\psi_N - \psi')/\tau_r} dt'.$$

$$F_{cdN}^*(r) = \int_0^{t_{N-1}} e^{-(\psi_N - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt'. \quad (36)$$

Note that  $J1_N^*(r)$  is the same quantity that appears in the evaluation of  $x_{cdN}$ . Using equation (33) and (34), equation (32) is rewritten as,

$$\gamma_{cdN} = G_{cdN} n_N + K_{cdN}. \quad (37)$$

A recursive relationship for equation (37) is developed in the Appendix and is given below as:

$$F_{cdN}^*(r) = e^{-\frac{1}{\tau_r} \Delta\psi_N} \left\{ F_{cdN-1}^*(r) + (G_{cdN-1} - G_{cdN-2}) J1_{N-1}^*(r) \right\} \quad (38)$$

where  $F_{cdN-1}^*(r)$  and  $J1_{N-1}^*(r)$  are known from the previous time step. Thus  $\gamma_{cdN}$  in equation (37), can be evaluated by integrating  $J1_N^*(r)$  given in equation (17) over the current time step only using numerical quadrature as outlined in the previous section. All history effects are represented by  $K_{cdN}$  given by equation (34).

#### INCREMENTAL EQUATION OF EQUILIBRIUM

In this section, an incremental finite element formulation for the isothermal, elastic-plastic-creep-large strain problem is developed. The formulation is suitable for use with any creep constitutive theory which will predict the incremental creep strain component based on the creep strain rate at the beginning of the step. Although the creep and plasticity relations are for small strains, the formulation is developed for large strain kinematics. One may begin with the equations of equilibrium written in terms of the second Piola-Kirchhoff stress:

$$\frac{\partial}{\partial a_j} [S_{jk} (\delta_{ik} + \frac{\partial u_i}{\partial a_k})] + \rho_0 F_{0j} = 0 \quad (39)$$

where  $a_j$  and  $u_i$  are Langrangian coordinates and displacements, respectively.  $S_{jk}$  is the 2nd Piola-Kirchhoff stress tensor,  $\rho_0$  is undeformed density and  $F_{0j}$  is the body force per unit undeformed volume per unit mass. Applying the virtual work principle at time  $t+\Delta t$  yields

$$\int_{V_0} S_{ij}^{t+\Delta t} \delta E_{ij}^{t+\Delta t} dv = \delta R^{t+\Delta t} \quad (40)$$

where

$$\delta R^{t+\Delta t} = \int_S T_i^{t+\Delta t} s u_i^{t+\Delta t} ds + \int_{V_0} \rho_0 F_{0j}^{t+\Delta t} s u_i^{t+\Delta t} dv \quad (41)$$

and  $\delta E_{ij}^{t+\Delta t}$  is the variation of the Green-Lagrange strains at time  $t+\Delta t$  and  $T_{ij}^{t+\Delta t}$  are surface tractions at time  $t+\Delta t$  applied to the deformed surface  $S$ . Equation (40) may be put into incremental form by writing

$$\begin{aligned} S_{ij}^{t+\Delta t} &= S_{ij}^t + \Delta S_{ij} \\ E_{ij}^{t+\Delta t} &= E_{ij}^t + \Delta E_{ij} \end{aligned} \quad (42)$$

where  $S_{ij}^t$  and  $E_{ij}^t$  are stresses and strains at time  $t$  and  $\Delta S_{ij}$  and  $\Delta E_{ij}$  are increments of stress and strain, respectively. The strain increment may be decomposed into components which are linear and nonlinear in the displacement increments

$$\Delta E_{ij} = \Delta E_{ij}^L + \Delta E_{ij}^{NL} \quad (43)$$

where

$$\begin{aligned} \Delta E_{ij}^L &= \frac{1}{2} \left( \frac{\partial \Delta u_i}{\partial a_j} + \frac{\partial \Delta u_j}{\partial a_i} + \frac{\partial \Delta u_k}{\partial a_i} \frac{\partial u_k^t}{\partial a_j} + \frac{\partial u_k^t}{\partial a_i} \frac{\partial \Delta u_k}{\partial a_j} \right) \\ \Delta E_{ij}^{NL} &= \frac{1}{2} \left( \frac{\partial \Delta u_k}{\partial a_i} \frac{\partial \Delta u_k}{\partial a_j} \right) \end{aligned} \quad (44)$$

Substituting equations (42) and (43) into (40) yields

$$\begin{aligned} &\int_{V_0} S_{ij}^t \delta \Delta E_{ij}^L dv + \int_{V_0} S_{ij}^t \delta \Delta E_{ij}^{NL} dv \\ &+ \int_{V_0} \Delta S_{ij} \delta (\Delta E_{ij}^L + \Delta E_{ij}^{NL}) dv = \delta R^{t+\Delta t} \end{aligned} \quad (45)$$

The stress increment may be decomposed into two components, one which is dependent upon total strain and one which is independent of strain (i.e., creep, thermal, etc.):

$$\Delta S_{ij} = D_{ijkl} \Delta E_{kl} + \Delta P_{ij} \quad (46)$$

where  $D_{ijkl}$  is the usual effective tangent modulus and  $\Delta P_{ij}$  is a stress increment due to strain independent phenomena (as is usually assumed in creep). Substituting equation (46) into equation (45), making use of equation (43), and neglecting terms which would be nonlinear in displacement increments, yields the following:

$$\begin{aligned} &\int_{V_0} \Delta E_{kl}^L D_{ijkl} \delta (\Delta E_{ij}^L) dv \\ &+ \int_{V_0} (S_{ij}^t + \Delta P_{ij}) \delta (\Delta E_{ij}^{NL}) dv \\ &- \int_{V_0} (S_{ij}^t + \Delta P_{ij}) \delta (\Delta E_{ij}^L) dv + \delta R^{t+\Delta t} \end{aligned} \quad (47)$$

The term  $\Delta P_{ij}$  may be interpreted as the change in stress required to account for the creep and thermal strains. Equation (47) takes on the following form when put into matrix form

$$[M] (\ddot{q}^{t+\Delta t}) + ([K_L^t] + [K_{NL}^t]) (\Delta q) = (R^{t+\Delta t}) - (F^t) \quad (48)$$

where  $[M]$  is the mass matrix,  $[K_L^t]$  and  $[K_{NL}^t]$  are "linear" and "nonlinear" stiffness matrices,  $(R^{t+\Delta t})$  is a vector of forces due to externally applied loads,  $(F^t)$  is a vector of forces due to internal stress, and  $(\Delta q)$  is the increment of the nodal displacements. Complete details of the derivation of the quantities in equation (48), without creep, may be found in reference (17).

The determination of  $D_{ijkl}$  and  $\Delta P_{ij}$  depends on the specific hardening rule chosen. For simplicity of presentation, the isothermal model is chosen and it is assumed that the yield function can be expressed by

$$F = f - k^2 = 0. \quad (49)$$

For kinematic hardening,

$$f = \frac{1}{2} (S_{ij}^t - a_{ij}^t) (S_{ij}^t - a_{ij}^t) \quad (50)$$

where  $S_{ij}^t$  and  $a_{ij}^t$  are deviatoric components of the 2nd Piola-Kirchhoff stress and yield surface center, respectively and  $K$  is a constant.

The associated flow rule is given by

$$dE_{ij}^P = \lambda \frac{\partial F}{\partial S_{ij}} \quad (51)$$

where  $\lambda$  is a scalar to be determined. Using the Ziegler modification of the Prager kinematic work-hardening rule, it is assumed that the projection of the increment of stress onto the normal to the yield surface is a scalar multiple of the dot product of the plastic strain and yield surface normal:

$$dS_{ij} \frac{\partial F}{\partial S_{ij}} = C dE_{ij}^P \frac{\partial F}{\partial S_{ij}} \quad (52)$$

where  $C$  is a scalar (hardening modulus) to be determined from a uniaxial stress-strain curve. It can be shown that equation (52) is equivalent to the consistency condition for kinematic hardening and isothermal conditions. For small strains, a decomposition of strain is assumed:

$$dS_{ij} = E_{ijmn} (dE_{mn} - dE_{mn}^P - dE_{mn}^C) \quad (53)$$

where  $E_{ijmn}$  is the elastic constitutive tensor and  $dE_{mn}^C$  are the incremental creep strain components. The incremental creep strain components are determined from equation (8) for a time increment over which it is assumed the creep strain components are constant. Substituting equation (51) into equation (52) and (53) and solving the resulting equation for  $\lambda$  gives

$$\lambda = \frac{E_{ijmn} (dE_{mn} - dE_{mn}^C) \frac{\partial F}{\partial S_{ij}}}{E_{ijmn} \frac{\partial F}{\partial S_{mn}} \frac{\partial F}{\partial S_{ij}} + C \frac{\partial F}{\partial S_{ij}} \frac{\partial F}{\partial S_{ij}}} \quad (54)$$

Substituting equation (54) into equation (51) and the results into equation (53) and comparing the results with equation (46), one can show that the instantaneous modulus tensor is given by

$$D_{ijmn} = E_{ijmn} - \frac{E_{ijvw} \frac{\partial F}{\partial S_{vw}} \frac{\partial F}{\partial S_{tu}} E_{tumn}}{E_{pqrs} \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{rs}} + C \frac{\partial F}{\partial S_{pq}} \frac{\partial F}{\partial S_{pq}}} \quad (55)$$

and the change of stress due to creep strains is given by

$$dP_{ij} = - D_{ijmn} dE_{mn}^C. \quad (56)$$

Equation (53) can now be rewritten as,

$$dS_{ij} = D_{ijkl} (dE_{mn} - dE_{mn}^C). \quad (57)$$

#### DETERMINATION OF MATERIAL FUNCTIONS FROM EXPERIMENTAL DATA

In this section, equation (1) is specialized to the case of uniaxial loading; and to materials whose elastic response on loading and unloading in a creep and recovery test are the same. The required material functions can then be evaluated by a graphical procedure from the results of creep tests. For a general description of the methods for obtaining the material functions from experimental data, see Lou and Schapery (13).

Consider a uniaxial bar which is in a uniform uniaxial state of stress at a uniform temperature. Considering only one stress  $\sigma$ , and one strain  $\epsilon$ , we let  $\epsilon = \epsilon_{11}$  and  $\sigma = \sigma_{11}$ . When equation (1) is specialized to this uniaxial loading, the creep term is given by

$$\epsilon^C = g_1(\sigma) \int_0^t \Delta D(\psi - \psi') \frac{d(g_2\sigma)}{d\tau} d\tau \quad (58)$$

and the three independent material functions are given by

$$g_1(\sigma) = \frac{d\sigma_{11}}{d\sigma_{11}}, \quad (59)$$

$$g_2(\sigma) = G_{11}(\sigma) = \frac{\sigma_{11}}{\sigma_{11}}, \quad (60)$$

$$a_\sigma(\sigma) = \frac{a_D(\sigma)}{a_G(\sigma)}, \quad (61)$$

and

$$\Delta D(\psi) = \sum_r A(r) (1 - e^{-\psi/\tau_r}) + \sum_{r'} B(r') \psi. \quad (62)$$

It is shown in reference (12) that  $g_1(\sigma) = 1$  implies that the amount of elastic straining on unloading is the same as on loading and that  $\dot{\sigma}_{ab} = \sigma_{ab}$ . Using  $g_1(\sigma) = 1$  and  $\dot{\sigma}_{ab} = \sigma_{ab}$ , equation (58) is now given by,

$$\epsilon^C = \int_{-0}^t \Delta D(\psi - \psi') \frac{d(g_2\sigma)}{d\tau} d\tau \quad (63)$$

where

$$g_2\sigma = \frac{\sigma}{a_G}, \quad (64)$$

and

$$a_\sigma(\sigma) = \frac{a_D(\sigma)}{a_G(\sigma)}. \quad (65)$$

When equation (63) is subjected to a creep test, the resulting equation can be used to evaluate the material functions  $g_2$  and  $a_\sigma$ . Consider equation (63) subject to the stress history:

$$\sigma(t) = H(t) \sigma. \quad (66)$$

Substituting equation (66) into equation (63) gives

$$\epsilon^C = g_2 \Delta D \left( \frac{t}{a_G} \right) \sigma; \quad (67)$$

and the creep compliance is given by

$$\Delta D(t) = \frac{\epsilon^C}{\sigma} = g_2 \Delta D \left( \frac{t}{a_\sigma} \right). \quad (68)$$

Taking the log of both sides of equation (68) gives,

$$\log \Delta D(t) = \log g_2 + \log \Delta D \left( \frac{t}{a_\sigma} \right). \quad (69)$$

The simple form of equation (69) is such that a simple graphical procedure can be employed to evaluate the material functions. This is easily demonstrated by plotting  $\Delta D$  and  $t$  on double - logarithmic paper. Curves at different stress levels can be superimposed by translating them along the  $\Delta D$  and  $t$  axes. If the curves are shifted to a reference stress,  $\sigma_r$ , the amount of horizontal shift ( $t$ ) and vertical shift ( $\Delta D$ ) equals  $\log g_2 = \log \frac{g_2(\sigma)}{g_2(\sigma_r)}$  and  $\log a_\sigma = \log \frac{a_\sigma(\sigma)}{a_\sigma(\sigma_r)}$ , respectively. These shifts are graphically demonstrated in Figure (1).

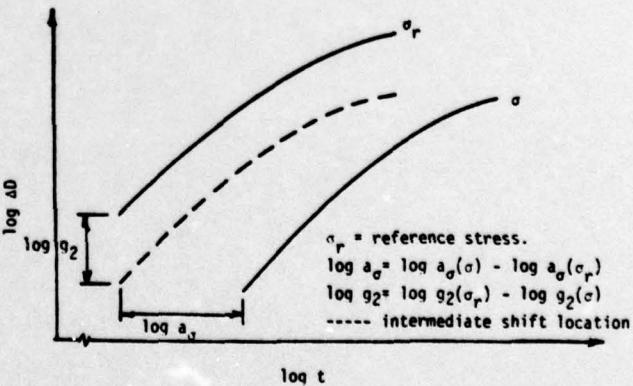


Fig. 1 Horizontal and Vertical Shift of Creep Compliance

The direction of the shifts are as follows:

$$\frac{g_2(\sigma)}{g_2(\sigma_r)} < 1, \text{ shift is up.}$$

$$\frac{g_2(\sigma)}{g_2(\sigma_r)} > 1, \text{ shift is down,}$$

and

$$\frac{a_0(\sigma_r)}{a_0(\sigma)} < 1, \text{ shift is to left,}$$

$$\frac{a_0(\sigma_r)}{a_0(\sigma)} > 1, \text{ shift is to right.}$$

### SOME NUMERICAL RESULTS

This section presents some numerical results obtained for a type 304 stainless steel with the computer program NONVIS developed by the authors which implements the procedures developed above for integrating equation (1) for the following stress histories: creep (Figure 2), creep-recovery (Figure 3), multi-step (Figure 4), and stress reversal (Figure 5). The results obtained are compared with those of the phenomenological creep theory based on strain hardening using total creep strain (18).

An effective creep equation of the form:

$$\dot{\epsilon}^c = A(1 - e^{-rt}) + kt \quad (70)$$

was used to determine the material functions in lieu of the actual experimental data. Two different heats of type 304 stainless steel are considered. The creep strain parameters for heat no. 9T2796 at 1100F are taken as:

$$A(\bar{\sigma}) = 5.436 \times 10^{-5} \bar{\sigma}^{1.843} \quad (71a)$$

$$r(\bar{\sigma}) = 5.929 \times 10^{-5} \exp(0.2029 \bar{\sigma}) \quad (71b)$$

$$k(\bar{\sigma}) = 6.73 \times 10^{-9} [\sinh(0.1479 \bar{\sigma})]^{3.0} \quad (71c)$$

with  $\bar{\sigma}$  (effective stress) in ksi, and  $t$  in hours. The creep strain parameters for heat No. 8043813 at 1200F are taken as

$$A(\bar{\sigma}) = 2.33 \times 10^{-6} \bar{\sigma}^{3.083} \quad (72a)$$

$$r(\bar{\sigma}) = 1.354 \times 10^{-3} \exp(0.129 \bar{\sigma}) \quad (72b)$$

and

$$k(\bar{\sigma}) = 7.91 \times 10^{-11} [\sinh(0.1932 \bar{\sigma})]^{4.0} \quad (72c)$$

The vertical shift is defined by the function  $A(\sigma)$  and the horizontal shift by the function  $r(\bar{\sigma})$ . This can be seen by equating the transient term given in equation (58) (for  $r = 1$ ,  $r' = 0$ ) and (70) from which it may be shown that

$$A = g_2(\sigma) \quad (73a)$$

$$\dot{\epsilon} = rt \quad (73b)$$

$$A^{(1)} = g_2(\sigma_r) \quad (73c)$$

and

$$r_1 = 1/a_0(\sigma_r) \quad (73d)$$

Rearranging equation (73a) gives

$$A/\sigma = g_2 \quad (74)$$

and from equation (2a) and (73b)

$$r = 1/a_0 \quad (75)$$

Therefore, the actual shifting of the creep compliance is not necessary to determine the material functions  $g_2(\sigma)$  and  $\frac{1}{a_0(\sigma)}$ . It is easy to validate the function  $g_2$  and  $\frac{1}{a_0}$  given above by using the graphical procedure.

If the steady state creep term in equation (70), i.e., the term linear in  $t$ , is subjected to the creep stress history given by equation (66), one obtains

$$\epsilon_s = B^{(1)} \frac{g_2(\sigma)}{a_0} t. \quad (76)$$

Using equations (74) and (75), equation (76) becomes

$$\epsilon_s = B^{(1)} A r t. \quad (77)$$

Equation (70) above gives for the steady state creep term

$$\epsilon_s = kt. \quad (78)$$

However, from equations (77), (78), and (71c) or (72c), it is seen that equations (77) and (78) are not equal. Thus it is concluded that a single reduced time cannot be used to relate both the transient and steady state creep terms using the creep equations given by either equation (71) or (72). Hence, in the present application to the type 304 stainless steels, the steady state term will be taken as the second term of equation (70),  $\epsilon_s = kt$ .

Figure 2 presents creep strain vs. time for a uniaxial stainless steel specimen (heat No. 9T2796) for a creep test. The results of both the nonlinear viscoelastic and the phenomenological creep theory are nearly identical as is expected. Thus, only the nonlinear viscoelastic theory is presented.

Figure 3 presents the creep strain response of a type 304 stainless steel (heat No. 8043813) uniaxial specimen subjected to a creep and recovery test. The results indicate very little recovery; and, of course, the phenomenological theory using strain hardening predicts no recovery. However, it should be noted that the nonlinear viscoelastic theory would predict complete recovery if the creep-recovery curve were extended to larger values of time.

Figure 4 presents the creep strain response of a type 304 stainless steel (heat No. 8043813) uniaxial specimen subjected to monotonically increasing load. The results are in reasonable agreement with the experimental results and are very close to the results based on strain hardening.

Figure 5 presents the creep strain response of a uniaxial specimen subjected to a reversed cyclic

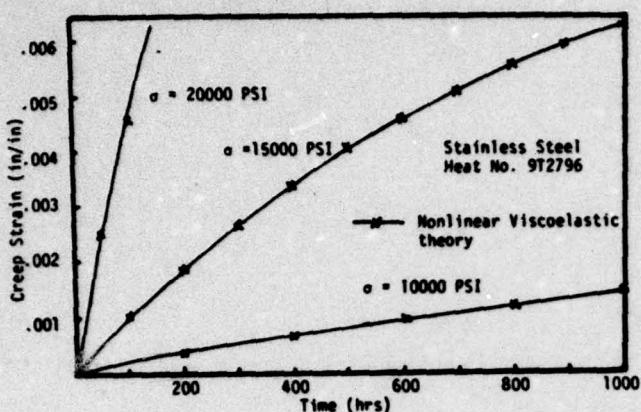


Fig. 2 Creep of Uniaxial Specimen

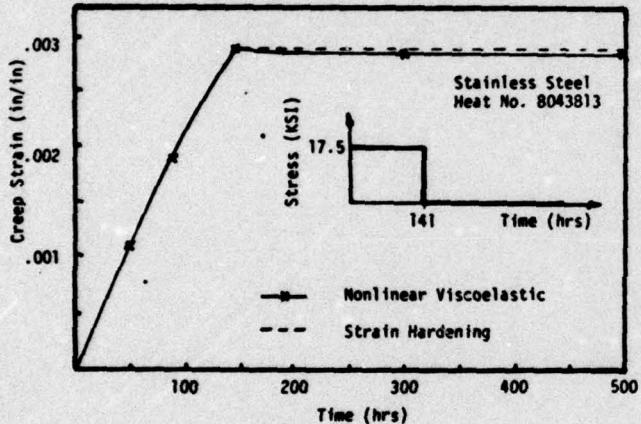


Fig. 3 Creep-Recovery of Uniaxial Specimen

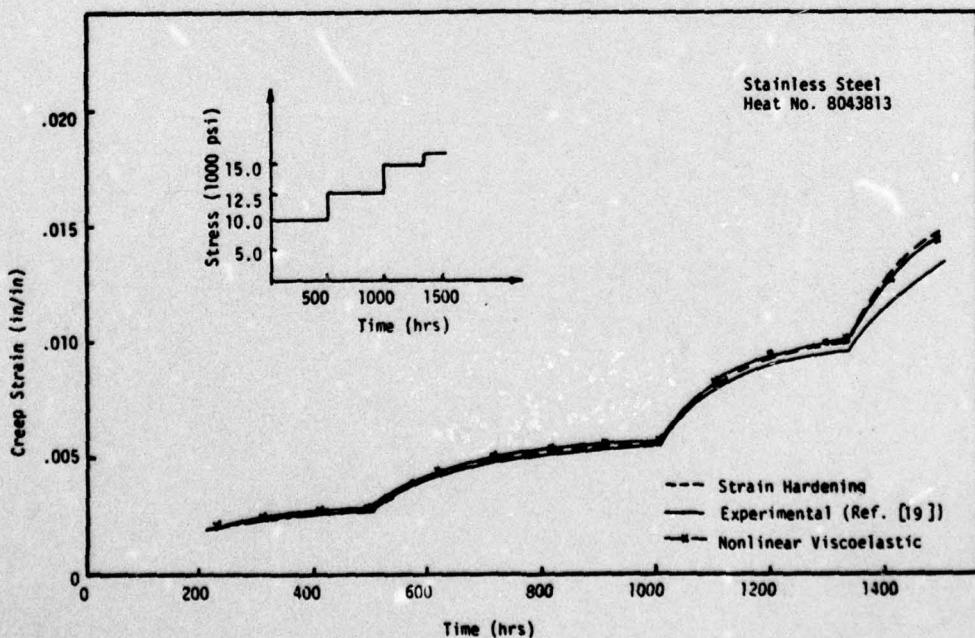


Fig. 4 Creep Strain Response of Uniaxial Specimen Subjected to Monotonically Increasing Load

loading history. On the first reversal, the nonlinear viscoelastic theory over predicts the creep response as given by the experimental data. This is due in part to the lack of significant recovery for Type 304 stainless steel. To amplify this behavior, consider the stress history up to the time of the second stress reversal,

$$\sigma(t) = \sigma H(t) - 2\sigma H(t-t_1), \quad 0 < t < t_2 \quad (79)$$

Substituting equation (79) into equation (63) gives

$$\epsilon^c = D(\psi) g_2 \sigma - 2g_2 \sigma D(\psi-\psi_1). \quad (80)$$

Thus it is seen that the second term gives  $2g_2 \sigma$  times the compliance. Thus twice the value of  $g_2 \sigma$  is multiplying the compliance than was on initial loading. This seems reasonable for materials with an active recovery, i.e., polymers. However, for metals the recovery is only slight and the second term over predicts its contribution to the recovery. Further, the first term is related to the concept of a fading memory, which is questionable for the stress reversal situation. Note the curve given for the first stress reversal with no recovery or memory of past history. This curve was obtained by initializing equation (20) to zero, which represents the memory of the material to prior histories, and setting  $G_{11N-2} = 0$  when a stress reversal occurs. This gives the same response as an initial loading with the strain accumulated at the time of the reversal serving as a new origin; this is the current procedure being followed in reference (14) for the first stress reversal using the phenomenological theory.

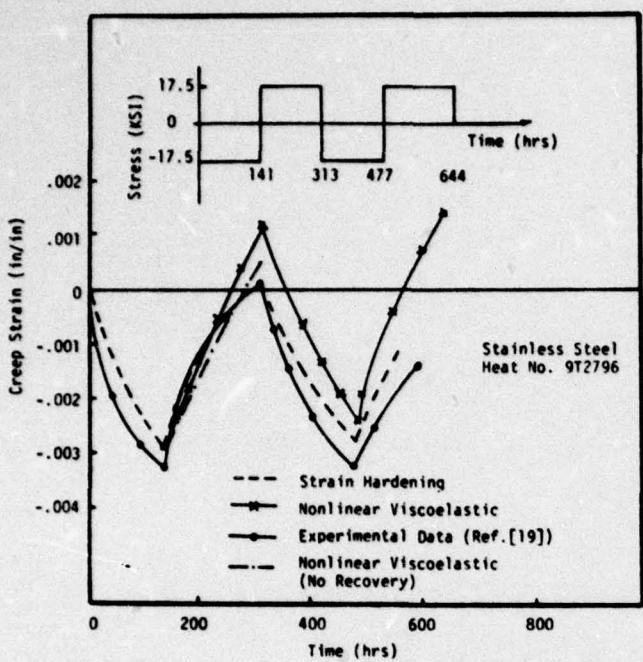


Fig. 5 Creep Strain Response of Uniaxial Specimen Subjected to Cyclic Load

#### CLOSURE

The purpose of this paper was first to present an efficient method of integrating Schapery's single integral equation for use in an incremental finite element program wherein the equations of motion are integrated step-by-step. And secondly to present some preliminary results for metals using this theory.

It is believed that the method presented for integrating the single integral equation is efficient based on the fact that the integrations required are for the current time step only and does not require integration over the entire past history to determine the current creep rate. From the experience gained with the computer program NONVIS, the programming aspect of implementing the theory is straight forward and can easily be incorporated into existing finite element programs.

The results of this paper are not presented to draw final conclusions regarding whether Schapery's nonlinear viscoelastic theory or the phenomenological theory based on strain hardening is superior. Obviously, the application of the phenomenological theory for metals has developed over many years whereas the application of nonlinear viscoelastic theory to metals is relatively new. However, from the results presented here, it is seen that in all except the case of stress reversal, the nonlinear viscoelastic theory gives results as acceptable as those based on strain hardening theory and in fair agreement with experimental data. For the stress reversal case, two problems arise. The nonlinear viscoelastic theory predicts a complete recovery of the strain accumulated at the first and subsequent reversals (as in linear viscoelasticity). Secondly, the theory has a fading memory of past stress history. However, as was noted, these effects can be neglected for the stress reversal case by initializing to zero the parameters that

contain the memory of past histories when a stress reversal occurs and setting the value of  $G_{11}^{N-2} = 0$ .

Although the results obtained herein have been for the response of a stainless steel, most of the results reported to date have been for a polymer or polymer composite. As noted in the introduction, references (12,13) show the theory to be in good agreement for a polymer composite for the following stress histories: creep, creep-recovery, and multi-step inputs. Therefore, it is believed that the use of the integration technique for the single integral equation in conjunction with an incremental finite element analysis has immediate application.

Current work with the single integral equation of Schapery's involves: (1) providing a better predictive capability for the stress reversal and other stress histories for metals, (2) determining the creep compliance and material function for several composite materials and (3) incorporating the above integration procedure into a finite element program for testing on two dimensional problems.

#### ACKNOWLEDGEMENT

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## APPENDIX

### Recursive Relationships for $C1_{cd_N}^{(r)}$ and $C2_{cd_N}^{(r')}$

After expanding equation (23) as the sum of two integrals, it can be rewritten as

$$C1_{cd_N}^{(r)} = e^{-(\psi_N - \psi_{N-1})/\tau_r} \times \left\{ \int_0^{t_{N-2}} e^{-(\psi_{N-1} - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' + \int_{t_{N-2}}^{t_{N-1}} e^{-(\psi_{N-1} - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' \right\}. \quad (81)$$

To evaluate equation (81), note that if in equation (23) N is replaced by N-1, equation (23) can be written as,

$$C1_{cd_{N-1}}^{(r)} = \int_0^{t_{N-2}} e^{-(\psi_{N-1} - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' \quad (82)$$

which is the value of  $C1_{cd}^{(r)}$  from the previous time step.

Define  $\Delta\psi_N$  as,

$$\Delta\psi_N = \psi_N - \psi_{N-1} \quad (83)$$

and approximate the time rate of change of  $G_{cd}$  by

$$\frac{dG_{cd}}{dt} = \frac{G_{cd_{N-1}} - G_{cd_{N-2}}}{\Delta t_{N-1}} \quad (84)$$

on the time interval  $t_{N-2} < t < t_{N-1}$ . Using equation (82), (83) and (84) in equation (81) gives,

$$C1_{cd_N}^{(r)} = e^{-\Delta\psi_N/\tau_r} \left\{ C1_{cd_{N-1}}^{(r)} + \frac{G_{cd_{N-1}} - G_{cd_{N-2}}}{\Delta t_{N-1}} \int_{t_{N-2}}^{t_{N-1}} e^{-(\psi_{N-1} - \psi')/\tau_r} dt' \right\}. \quad (85)$$

In the integral given by equation (17), replace N by N-1 to get,

$$J1_{N-1}^{(r)} = \frac{1}{\Delta t_{N-1}} \int_{t_{N-2}}^{t_{N-1}} e^{-(\psi_{N-1} - \psi')/\tau_r} dt' \quad (86)$$

which is  $J1_{N-1}^{(r)}$  from the previous step. Thus, equation (85) can be written as,

$$C1_{cd_N}^{(r)} = e^{-\Delta\psi_N/\tau_r} \left\{ C1_{cd_{N-1}}^{(r)} + (G_{cd_{N-1}} - G_{cd_{N-2}}) J1_{N-1}^{(r)} \right\}. \quad (87)$$

Note the values of  $C1_{cd_{N-1}}^{(r)}$ ,  $J1_{N-1}^{(r)}$ ,  $G_{cd_{N-1}}$ , and  $G_{cd_{N-2}}$  are known values from previous steps.

Similarly, equation (24) can be manipulated and rewritten as

$$C2_{cd_N}^{(r')} = \Delta\psi_N G_{cd_{N-1}} + C2_{cd_{N-1}}^{(r')} + (G_{cd_{N-1}} - G_{cd_{N-2}}) \times J2_{N-1}^{(r')} \quad (88)$$

where  $C2_{cd_{N-1}}^{(r')}$  and  $J2_{N-1}^{(r')}$  are values of equation (24) and (18) from the previous step.

### Recurvise Relationship for $F_{cd_N}^{(r)}$

After expanding equation (36) as the sum of two integrals, it can be written as,

$$F_{cd_N}^{(r)} = \int_0^{t_{N-2}} e^{-(\psi_N - \psi_{N-1} + \psi_{N-1} - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' + \int_{t_{N-2}}^{t_{N-1}} e^{-(\psi_N - \psi_{N-1} + \psi_{N-1} - \psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' \quad (89)$$

To evaluate equation (89), note that if in equation (36) N is replaced by N-1, equation (36) can be re-written as,

$$F_{cd_{N-1}}(r) = \int_0^{t_{N-2}} e^{-(\psi_{N-1}-\psi')/\tau_r} \frac{dG_{cd}}{dt'} dt' \quad (90)$$

which is the value of  $F_{cd}(r)$  from the previous step. Using equation (89), (90) and (84), equation (36) can be written as,

$$F_{cd_N}(r) = e^{-\Delta\psi_N/\tau_r} \left\{ F_{cd_{N-1}}(r) + \frac{G_{cd_{N-1}} - G_{cd_{N-2}}}{\Delta t_{N-1}} \times \int_{t_{N-2}}^{t_{N-1}} e^{-(\psi_{N-1}-\psi')/\tau_r} dt' \right\}. \quad (91)$$

The integral term in equation (91) can be written as  $J1_N(r)$  using equation (86) to obtain

$$F_{cd_N}(r) = e^{-\Delta\psi_N/\tau_r} \left\{ F_{cd_{N-1}}(r) + (G_{cd_{N-1}} - G_{cd_{N-2}}) J1_N(r) \right\}. \quad (92)$$

Evaluation of the Quantities  $J1_N(r)$  and  $J2_N(r')$   
Expand  $J1_N(r)$  in equation (17) as

$$J1_N(r) = \frac{1}{\Delta t_N} \left\{ \int_{t_{N-1}}^{t_N} e^{-(\psi_N-\psi')/\tau_r} dt' + \int_{t_{N-1}}^{t_N} e^{-\frac{a}{\tau_r}(t_N-t')} dt' - \int_{t_{N-1}}^{t_N} e^{-\frac{a}{\tau_r}(t_N-t')} dt' \right\}$$

or

$$J1_N(r) = \frac{1}{\Delta t_N} \left\{ \int_{t_{N-1}}^{t_N} e^{-\frac{a}{\tau_r}(t_N-t')} dt' + \int_{t_{N-1}}^{t_N} \left[ e^{-(\psi_N-\psi')/\tau_r} - e^{-\frac{a}{\tau_r}(t_N-t')} \right] dt' \right\}. \quad (93)$$

Let the change in reduced time during the interval  $t_N$  be related to  $t_N$  through the parameter,  $a$ , such that

$$\psi_N - \psi_{N-1} = a(t_N - t_{N-1}) \text{ or } a = \frac{\Delta\psi_N}{\Delta t_N}. \quad (94)$$

Defining the quantity  $J1_N^*(r)$  as

$$J1_N^*(r) = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} \left[ e^{-(\psi_N-\psi')/\tau_r} - e^{-\frac{a}{\tau_r}(t_N-t')} \right] dt' \quad (95)$$

allows equation (93) to be rewritten as

$$J1_N(r) = \frac{1}{\Delta t_N} \frac{\tau_r}{a} \left[ e^{-\frac{a}{\tau_r}(t_N-t)} \right]_{t_{N-1}}^{t_N} + J1_N^*(r) = \frac{\tau_r}{\Delta\psi_N} (1 - e^{-\Delta\psi_N/\tau_r}) + J1_N^*(r). \quad (96)$$

Making the following change of variable in  $J1_N^*(r)$

$$\tau = \frac{t'-t_N}{\Delta t_N} + \frac{t'-t_{N-1}}{\Delta t_N} = \frac{2t'-t_N-t_{N-1}}{\Delta t_N}$$

transforms the lower and upper limits of the integral in equation (93) to -1 and +1, respectively, and gives  $dt' = (\Delta t_N/2)d\tau$ .

Hence

$$J1_N^*(r) = \frac{1}{2} \int_{-1}^{+1} \left[ e^{-(\psi_N-\psi')/\tau_r} - e^{-\frac{a}{\tau_r}(t_N - \frac{\tau\Delta t_N + t_N + t_{N-1}}{2})} \right] d\tau$$

or

$$J1_N^*(r) = \frac{1}{2} \int_{-1}^{+1} \left[ e^{-(\psi_N-\psi')/\tau_r} - e^{-\frac{1}{\tau_r}(1-\tau)} \right] d\tau \quad (97)$$

where

$$\Delta\psi_N = \psi_N - \psi_{N-1}.$$

Equation (97) can be integrated numerically by standard Gaussian quadrature.

Expanding  $J2_N(r')$  in equation (18) as

$$J2_N(r') = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} (\psi_N - \psi') dt'. \quad (98)$$

Upon integration of the first term, equation (98) becomes

$$J2_N(r') = \psi_N - \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} \psi' dt'. \quad (99)$$

Define,

$$J2_N^*(r') = \frac{1}{\Delta t_N} \int_{t_{N-1}}^{t_N} \psi' dt' \quad (100)$$

and write equation (99) as

$$J2_N(r') = \psi_N - J2_N^*(r') \quad (101)$$

Using the same change of variable as above allows equation (100) to be rewritten as

$$J2_N^*(r') = \frac{1}{2} \int_{-1}^{+1} \psi' d\tau \quad (102)$$

#### Evaluation of $\psi$

Recalling that the reduced time is defined by

$$\psi = \int_0^t \frac{a_G}{a_D} dt' ;$$

for the interval  $t_{N-1} \leq t \leq t_N$ , one can then write

$$\psi = \int_0^{t_{N-1}} \frac{a_G}{a_D} dt' + \int_{t_{N-1}}^{t_N} \frac{a_G}{a_D} dt'$$

or

$$\psi = \psi_{N-1} + \int_{t_{N-1}}^{t_N} \frac{a_G}{a_D} dt' \quad (103)$$

For small time intervals, one can approximate the stress as a linear function of time with the interval  $\Delta t_N$  such that

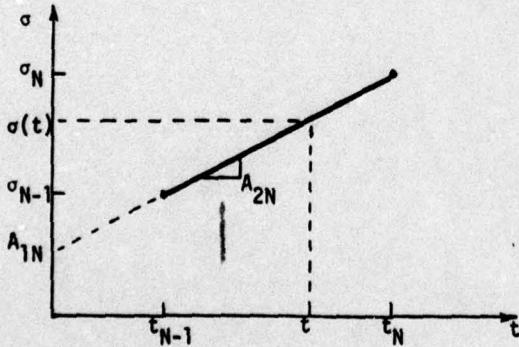


Fig. 6 Linear Interpolation of Stress

from which

$$\sigma(t) = A_{1N} + A_{2N}t \quad , \quad t_{N-1} \leq t \leq t_N$$

where

$$A_{1N} = \frac{\sigma_{N-1} t_N - \sigma_N t_{N-1}}{\Delta t_N} \quad \text{and} \quad A_{2N} = \frac{\Delta \sigma_N}{\Delta t_N}$$

Changing the variable of integration to  $\sigma$ , equation (103) becomes

$$\psi = \psi_{N-1} + \int_{\sigma_{N-1}}^{\sigma_N} \frac{a_G(\sigma')}{a_D(\sigma')} \frac{d\sigma'}{A_{2N}} \quad (104)$$